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Convergence of the Kato Approximants for Evolution Equations Involving Functional Perturbations

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The existence of a unique strong solution of the nonlinear abstract functional differential equation

$$u'(t) + A(t)\,u(t) = F(t, u_t), \qquad u_0 = \phi \in C^1([-r, 0], X), t \in [0, T], \tag{E}$$

is established. X is a Banach space with uniformly convex dual space and, for $t \in [0, T]$, $A(t)$ is m-accretive and satisfies a time dependence condition suitable for applications to partial differential equations. The function F satisfies a Lipschitz condition. The novelty of the paper is that the solution $u(t)$ of (E) is shown to be the uniform limit (as $n \to \infty$) of the sequence $u_n(t)$, where the functions $u_n(t)$ are continuously differentiable solutions of approximating equations involving the Yosida approximants. Thus, a straightforward approximation scheme is now available for such equations, in parallel with the approach involving the use of nonlinear evolution operator theory.

1. INTRODUCION

Let X denote a real Banach space with norm $\|\cdot\|$. Let $C = C([-r, 0], X)$, $0 < r < +\infty$, denote the space of continuous functions mapping $[-r, 0]$ into X. C is a Banach space with norm $\|\phi\|_C = \sup_{\theta \in [-r,0]} \|\phi(\theta)\|$ for $\phi \in C$. We consider the nonlinear abstract problem

$$u'(t) + A(t)\,u(t) = F(t, u_t), \qquad u_0 = \phi, t \in [0, T], \tag{E}$$

where $u\colon [-r, T] \to X$; for each $t \in [0, T]$, $A(t)\colon D(A(t)) \subset X \to X$; $F\colon [0, T] \times C \to X$, and $u_t \in C$ is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$. By a strong slution of (E) on $[0, T]$ we mean an absolutely continuous X-valued function which, for almost all $t \in [0, T]$, is strongly differentiable and satisfies (E).

The purpose of this paper is to establish, under certain additional assumptions on X and the operators A, F, the existence of a unique strong solution $u(t)$, $t \in [0, T]$, of (E) for each $\phi \in C^1 = C^1([-r, 0], X)$ (the space of X-valued continuously differentiable functions on $[-r, 0]$) with

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$\phi(0) \in D(t)$ ($=D$, independent of t). The solution, $u(t)$, will be shown to depend continuously on ϕ, in a sense to be made more precise later.

Specifically, we impose the following conditions:

(C.1) X^*, the dual space of X, is uniformly convex.

(C.2) The domain D of $A(t)$ is independent of t.

(C.3) There is a nondecreasing function $L: [0, \infty) \to [0, \infty)$ such that for all $x \in D$ and $s, t \in [0, T]$,

$$\|A(t)x - A(s)x\| \leqslant |t-s|\, L(\|x\|)(1 + \|A(s)x\|).$$

(C.4). For each $t \in [0, T]$, $A(t)$ is m-accretive (see below).

(C.5) There exists a constant $B > 0$ such that

$$\|F(t, \phi) - F(t, \psi)\| \leqslant B\, \|\phi - \psi\|_C, \phi, \psi \in C, t \in [0, T].$$

(C.6) There exists an increasing function $g: [0, \infty) \to [0, \infty)$ such that

$$\|F(t, \phi) - F(s, \phi)\| \leqslant |t-s|\; g(\|\phi\|_C), \phi \in C, s, t \in [0, T].$$

We recall the definition of a single-valued operator $A: D(A) \subset X \to X$ being m-accretive. Let $\langle x, y \rangle$ denote the evaluation $y(x)$ for $x \in X$, $y \in X^*$. Define

$$J(x) = \{x^* \in X^*; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The set $J(x)$ is nonempty for each $x \in X$ by the Hahn Banach theorem. The mapping J is called the *duality map* of X. For a general Banach space X, the duality map may be multi-valued. However, if X^* is strictly convex, then the duality map J is single-valued. If, moreover, X^* is uniformly convex, then J is uniformly continuous on bounded subsets on X. An operator $A: D(A) \subset X \to A$ is called *accretive* if for every $x_1, x_2 \in D(A)$ we have

$$\langle Ax_1 - Ax_2, J(x_1 - x_2) \rangle \geqslant 0.$$

An accretive operator A is said to be m-accretive if $R(I + \lambda A) = X$ for some $\lambda > 0$. If A is m-accretive, then $R(I + \lambda A) = X$ for all $\lambda > 0$ (see, for example, [13, Lemma 2.1]).

The functional differential equation (E) and its autonomous counterpart

$$u'(t) + \mathrm{Au}(t) = F(u_t), \qquad u_0 = \phi, t \in [0, T] \tag{AE}$$

have been the subject of considerable research activity during the past decade. Various conditions on X, A, and F have been considered, and existence results for different spaces Q of initial functions have been obtained. The main tool for the autonomous case has been the nonlinear semigroup theory. We mention here the work of Travis and Webb [16; linear A, $Q = C$], Webb [17; $A \equiv 0$, $Q = C$], [18; $A + \alpha I$ accretive for some $\alpha \in R$, $Q = C$], [19; $A + \alpha I$ accretive, $Q = L^p([-r, 0], H)$, H a Hilbert space], Brewer [1; $A \equiv 0$, $Q =$ fading memory type of space], [2; $A + \alpha I$ accretive, $Q =$ fading memory type of space], Flaschka and Leitman [10; $A \equiv 0$, $Q = C$], and Plant [15; $A \equiv 0$, $Q = C$ and $Q = L^p([-r, 0], X)$].

For the nonautonomous case, existence results have been obtained by means of the Crandall–Pazy nonlinear evolution operator theory [4]. For results in this direction we refer the reader to the papers of Fitzgibbon [7; linear A, $Q = C$], [8; linear A, $Q = C_{\text{unif}}((-\infty, 0], X)$], [9; linear A, $Q = C$ or C_{unif}], Webb [20; $A + \alpha I$ accretive, $Q = L^p([-r, 0], H)$], and Dyson and Bressan [5; $A + \alpha I$ accretive, $Q = C$], [6; $A + \alpha I$ accretive, $Q = C$]. The existence of solutions of (E) under the assumptions (C.1)–(C.6) follows from these two papers of Dyson and Bressan.

Our technique for proving the existence of solutions of (E) is different from the above. It does not involve the existence of a nonlinear evolution operator. Following Kato's approach in [13], we will show in a straightforward manner that $u(t)$, the unique strong solution of (E), actually exists as a uniform limit of $\{u_n(t)\}$, where $u_n(t)$, $n = 1, 2, \ldots$, are the unique strongly continuously differentiable solutions of the approximating equations

$$u_n'(t) + A_n(t)\, u(t) = F(t, u_{n_t}), \qquad u_{n_0} = \phi, \; t \in [0, T], \tag{E_n}$$

where $A_n(t)$ are the Yosida approximants.

For nonfunctional results in this direction the reader is referred to Kartsatos and Zigler [11] and Kartsatos [12].

2. The Results

For each $t \in [0, T]$, $n = 1, 2, \ldots$, the Yosida approximants $J_n(t)$ and $A_n(t)$ are defined as follows:

$$J_n(t) = (I + (1/n)\, A(t))^{-1}, \tag{2.1}$$

$$A_n(t) = n(I - J_n(t)). \tag{2.2}$$

If Condition (C.4) is satisfied, then the Yosida approximants are everywhere defined and

$$A_n(t) = A(t)\, J_n(t) = A(t)(I + (1/n)\, A(t))^{-1}.$$

For other properties of the operators $J_n(t)$, $A_n(t)$ which hold as a consequence of (C.1)–(C.4), see [13, Lemmas 2.2–2.5].

We henceforth consider Eqs. (E_n) with $A_n(t)$ as defined by (2.1) and (2.2). We now state our main result.

THEOREM 2.1. *Assume that Conditions* (C.1)–(C.6) *hold. Then for every* $\phi \in C^1$ *with* $\phi(0) \in D$ *there exists a unique strong solution* $u(t)$ *of* (E) *on* $[0, T]$ *given by* $\lim_{n\to\infty} u_n(t)$, *where, for each* n, $n = 1, 2,\ldots, u_n(t)$ *is the unique continuously differentiable solution of* (E_n) *on* $[0, T]$.

Proof. The proof of Theorem 2.1 is accomplished by a series of lemmas. We first verify that for each n, $n = 1, 2,\ldots,$ Eq. (E_n) has a unique continuously differentiable solution $u_n(t)$. Then the uniform boundedness of $\{u_n(t)\}$ and $\{u_n'(t)\}$ is established. Finally, we show that the strong limit $u(t) = \lim_{n\to\infty} u_n(t)$ exists uniformly on $[0, T]$, $u(t)$ is uniformly Lipschitz continuous on $[0, T]$ with $u_0 = \phi$, and the strong derivative of $u(t)$ exists almost everywhere on $[0, T]$, and equals $-A(t)\, u(t) + F(t, u_t)$.

LEMMA 2.2. *Assume that Conditions* (C.1)–(C.6) *hold. Then for every* $\phi \in C^1$ *with* $\phi(0) \in D$ *there exists a unique strongly continuously differentiable solution* $u_n(t)$ *of* (E_n) *on* $[0, T]$.

Proof. In a manner similar to that of [13, Lemma 4.1] (see also the notes at the end of [13]), it can be shown that for all n and $x \in X$ we have

$$\|A_n(t)x - A_n(s)x\| \leqslant |t - s|\, L_1(\|x\|)(1 + \|A_n(s)x\|), \tag{2.3}$$

where L_1 is a nondecreasing function. Inequality (2.3) shows that $A_n(t)x$ is Lipschitz continuous in t for every $x \in X$. Also, $A_n(t)$ is uniformly Lipschitz continuous in x for $t \in [0, T]$ ([13, Lemma 2.2]). Thus, there exists a unique strongly continuously differentiable solution, $u_n(t)$, of (E_n) on $[0, T]$ (see, for example, [14, Theorems 6.2.2 and 6.2.3]).

LEMMA 2.3. *Assume that Conditions* (C.1)–(C.6) *hold. Assume also that* $\phi(0) = a \in D$ *and* $\phi \in C^1$. *Then there exists* $K > 0$ *such that* $\|u_n(t)\| \leqslant K$ *for all* $n = 1, 2,\ldots,$ *and* $t \in [0, T]$, *where* $u_n(t)$ *are the solutions of* (E_n).

Proof. Since $u_n(t)$ is differentiable on $[0, T]$ and satisfies (E_n), we have

$$\begin{aligned}
&\langle u_n'(t), J(u_n(t) - a)\rangle \\
&\quad = -\langle A_n(t)\, u_n(t) - F(t, u_{n_t}), J(u_n(t) - a)\rangle \\
&\quad = -\langle A_n(t)\, u_n(t) - A_n(t)a, J(u_n(t) - a)\rangle \\
&\qquad - \langle A_n(t)a, J(u_n(t) - a)\rangle + \langle F(t, u_{n_t}), J(u_n(t) - a)\rangle \\
&\quad \leqslant -\langle A_n(t)a, J(u_n(t) - a)\rangle + \langle F(t, u_{n_t}), J(u_n(t) - a)\rangle \\
&\quad \leqslant -\langle A_n(t)a, J(u_n(t) - a)\rangle + \langle F(t, u_{n_t}) - F(t, \phi), J(u_n(t) - a)\rangle \\
&\qquad + \langle F(t, \phi), J(u_n(t) - a)\rangle \\
&\quad \leqslant \|A_n(t)a\|\, \|u_n(t) - a\| + B\, \|u_{n_t} - \phi\|_C\, \|u_n(t) - a\| + M_1\|u_n(t) - a\|,
\end{aligned}$$

where M_1 is an upper bound for $\|F(t, \phi)\|$. Here we have made use of the accretiveness of $A_n(t)$, (C.5), and the continuity of F in t. From (2.3) we obtain

$$\begin{aligned}\|A_n(t)a - A_n(0)a\| &\leqslant tL_1(\|a\|)(1 + \|A_n(0)a\|)\\ &\leqslant TL_1(\|a\|)(1 + \|A(0)a\|)\end{aligned}$$

[13, Lemma 2.3], which yields

$$\|A_n(t)a\| \leqslant TL_1(\|a\|)(1 + \|A(0)a\|) + \|A(0)a\| = K_1.$$

Thus,

$$\begin{aligned}\langle u_n'(t), J(u_n(t) - a)\rangle \leqslant K_1 \|u_n(t) - a\| &+ B\|u_{n_t} - \phi\|_C \|u_n(t) - a\|\\ &+ M_1 \|u_n(t) - a\|.\end{aligned} \tag{2.4}$$

Since $u_n(t)$ is strongly absolutely continuous, so is $\|u_n(t) - a\|$. Thus, $(d/dt)\|u_n(t) - a\|$ exists a.e. and, by [13, Lemma 1.3] and (2.4), we obtain

$$\begin{aligned}(d/dt)\|u_n(t) - a\| &\leqslant K_1 + B\|u_{n_t} - \phi\|_C + M_1\\ &= K_2 + B\|u_{n_t} - \phi\|_C,\end{aligned} \tag{2.5}$$

where $K_2 = K_1 + M_1$. Now we integrate (2.5) to obtain

$$\|u_n(t) - a\| \leqslant K_2 T + B\int_0^t \|u_{n_s} - \phi\|_C\, ds, \qquad t \in [0, T]. \tag{2.6}$$

Case 1. Suppose $t \geqslant r$. Then $t + \theta \geqslant 0$ for all $\theta \in [-r, 0]$. It follows that for such θ's, (2.6) implies

$$\begin{aligned}\|u_n(t+\theta) - a\| &\leqslant K_2 T + B\int_0^{t+\theta} \|u_{n_s} - \phi\|_C\, ds\\ &\leqslant K_2 T + B\int_0^t \|u_{n_s} - \phi\|_C\, ds.\end{aligned}$$

Since $\|u_n(t+\theta) - \phi(\theta)\| = \|u_n(t+\theta) - a + a - \phi(\theta)\|$, we have

$$\|u_n(t+\theta) - \phi(\theta)\| \leqslant \|a\| + \|\phi\|_C + K_2 T + B\int_0^t \|u_{n_s} - \phi\|_C\, ds.$$

Hence,

$$\sup_{\theta \in [-r,0]} \|u_n(t+\theta) - \phi(\theta)\| = \|u_{n_t} - \phi\|_C \leqslant \|a\| + \|\phi\|_C + K_2 T + B \int_0^t \|u_{n_s} - \phi\|_C \, ds. \tag{2.7}$$

Case 2. Suppose $0 \leqslant t < r$. Then, for $\theta \in [-r, -t)$, $t + \theta < 0$. For such θ's we obtain

$$\|u_n(t+\theta) - a\| = \|\phi(t+\theta) - a\| \leqslant \|\phi\|_C + \|a\|. \tag{2.8}$$

For $\theta \in [-t, 0]$, $t + \theta \geqslant 0$. Thus, (2.6) implies

$$\|u_n(t+\theta) - a\| \leqslant K_2 T + B \int_0^t \|u_{n_s} - \phi\|_C \, ds. \tag{2.9}$$

From (2.8) and (2.9) we conclude that for every $\theta \in [-r, 0]$ $(0 \leqslant t < r)$,

$$\|u_n(t+\theta) - a\| \leqslant \|a\| + \|\phi\|_C + K_2 T + B \int_0^t \|u_{n_s} - \phi\|_C \, ds,$$

from which it follows that

$$\|u_{n_t} - \phi\|_C \leqslant 2(\|a\| + \|\phi\|_C) + K_2 T + B \int_0^t \|u_{n_s} - \phi\|_C \, ds. \tag{2.10}$$

As a consequence of (2.7) and (2.10) we arrive at

$$\|u_{n_t} - \phi\|_C \leqslant 2(\|a\| + \|\phi\|_C) + K_2 T + B \int_0^t \|u_{n_s} - \phi\|_C \, ds \tag{2.11}$$

for every $t \in [0, T]$. An application of Gronwall's inequality in (2.11) shows the uniform boundedness of $\{\|u_{n_t} - \phi\|_C\}$. The uniform boundedness of $\{u_n(t)\}$ follows from $\|u_n(t) - a\| \leqslant \|u_{n_t} - \phi\|_C$.

Lemmas 2.4 and 2.5 will be needed in the proof of the uniform boundedness of $\{u_n'(t)\}$ (Lemma 2.6).

LEMMA 2.4. *Let $w \in C^1([0, T], X)$ be given. Then, for any $s \in [0, T)$,*

$$\lim_{h \to 0^+} \sup_{\theta \in [-s,0]} \|w(s+\theta+h) - w(s+\theta)\|/h$$

exists and equals

$$\sup_{\theta \in [-s,0]} \|w'(s+\theta)\|.$$

Proof. From

$$\sup_{\theta\in[-s,0]} \|w(s+\theta+h)-w(s+\theta)\|/h \geqslant \|w(s+\theta+h)-w(s+\theta)\|/h$$

for any $\theta \in [-s, 0]$, we obtain

$$\begin{aligned}\varliminf_{h\to 0^+} \sup_{\theta\in[-s,0]} &\|w(s+\theta+h)-w(s+\theta)\|/h\\ &\geqslant \varliminf_{h\to 0^+} \|w(s+\theta+h)-w(s+\theta)\|/h\\ &= \lim_{h\to 0^+} \|w(s+\theta+h)-w(s+\theta)\|/h\\ &= \|w'(s+\theta)\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\varliminf_{h\to 0^+} \sup_{\theta\in[-s,0]} &\|w(s+\theta+h)-w(s+\theta)\|/h\\ &\geqslant \sup_{\theta\in[-s,0]} \|w'(s+\theta)\|. \qquad (2.12)\end{aligned}$$

On the other hand, for $\theta \in [-s, 0]$, we find, for some $\lambda \in (0, 1)$,

$$\begin{aligned}\|w(s+\theta+h)&-w(s+\theta)\|/h\\ &\leqslant \|w'(s+\theta+\lambda h\|\, h/h = \|w'(s+\theta+\lambda h)\|\\ &\leqslant \sup_{\theta\in[-s+\lambda h,\lambda h]} \|w'(s+\theta)\|\\ &\leqslant \sup_{\theta\in[-s,\lambda h]} \|w'(s+\theta)\|.\end{aligned}$$

Hence,

$$\sup_{\theta\in[-s,0]} \|w(s+\theta+h)-w(s+\theta)\|/h \leqslant \sup_{\theta\in[-s,\lambda h]} \|w'(s+\theta)\|,$$

and

$$\begin{aligned}\varlimsup_{h\to 0^+} \sup_{\theta\in[-s,0]} &\|w(s+\theta+h)-w(s+\theta)\|/h\\ &\leqslant \varlimsup_{h\to 0^+} \sup_{\theta\in[-s,\lambda h]} \|w'(s+\theta)\|\\ &= \lim_{h\to 0^+} \sup_{\theta\in[-s,\lambda h]} \|w'(s+\theta)\|\\ &= \sup_{\theta\in[-s,0]} \|w'(s+\theta)\|. \qquad (2.13)\end{aligned}$$

Thus, from (2.12) and (2.13) we conclude that

$$\lim_{h\to 0^+} \sup_{\theta\in[-s,0]} \|w(s+\theta+h)-w(s+\theta)\|/h = \sup_{\theta\in[-s,0]} \|w'(s+\theta)\|.$$

LEMMA 2.5. *If* $w \in C^1([\bar{h},0],X)$ *and* $w \in C^1([0,\bar{h}],X)$ *for some* $\bar{h}>0$, *then*

$$\overline{\lim_{h\to 0^+}} \sup_{\theta\in[-(s+h),-s]} \|w(s+\theta+h)-w(s+\theta)\|/h \leqslant \|w'_+(0)\| + \|w'_-(0)\|$$

for $s \geqslant 0$, *where*

$$w'_+(t) = \overline{\lim_{h\to 0^+}} (w(t+h)-w(t))/h, \; w'_-(t) = \overline{\lim_{h\to 0^+}} (w(t)-w(t-h))/h.$$

Proof. We have

$$\begin{aligned}
&\sup_{\theta\in[-(s+h),-s]} \|w(s+\theta+h)-w(s+\theta)\|/h \\
&\quad = \|w(s+h+\theta_1)-w(s+\theta_1)\|/h \\
&\quad = \|w(s+h+(-s-\lambda h))-w(s+(-s-\lambda h))\|/h \\
&\quad = \|w((1-\lambda)h)-w(-\lambda h)\|/h \\
&\quad \leqslant \|w((1-\lambda)h)-w(0)\|/h + \|w(0)-w(-\lambda h)\|/h \\
&\quad \leqslant \|w'(\mu h)\| + \|w'(-\gamma h)\|.
\end{aligned}$$

Here $\theta_1 \in [-(s+h),-s]$, $\lambda \in [0,1]$, $\mu \in (0,1)$, and $\gamma \in (0,1)$. Thus,

$$\begin{aligned}
&\overline{\lim_{h\to 0^+}} \sup_{\theta\in[-(s+h,-s]} \|w(s+\theta+h)-w(s+\theta)\|/h \\
&\quad \leqslant \overline{\lim_{h\to 0^+}} \|w'(\mu h)\| + \overline{\lim_{h\to 0^+}} \|w'(-\gamma h)\| \\
&\quad = \|w'_+(0)\| + \|w'_-(0)\|.
\end{aligned}$$

LEMMA 2.6. *Assume that the conditions of Lemma* 2.3 *hold. Then there exists* $N>0$ *such that* $\|u_n'(t)\| \leqslant N$ *for all* $n=1,2,...,$ *and* $t\in[0,T]$. *Here* $u_n(t)$ *are the solutions of* (E_n).

Proof. Let $z_n(t) = u_n(t+h)-u_n(t)$ $(0<h<\min\{r,T\})$. Then, as in [13, Lemma 1.3],

$$\begin{aligned}
&\|z_n(t)\|\,(d/dt)\|z_n(t)\| \\
&\quad = \langle z_n'(t), J(z_n(t))\rangle \\
&\quad = -\langle A_n(t+h)\,u_n(t+h) - A_n(t)\,u_n(t), J(z_n(t))\rangle \\
&\qquad + \langle F(t+h, u_{n_{t+h}}) - F(t, u_{n_t}), J(z_n(t))\rangle \\
&\quad = -\langle A_n(t+h)\,u_n(t+h) - A_n(t+h)\,u_n(t), J(z_n(t))\rangle \\
&\qquad + \langle A_n(t)\,u_n(t) - A_n(t+h)\,u_n(t), J(z_n(t))\rangle \\
&\qquad + \langle F(t+h, u_{n_{t+h}}) - F(t+h, u_{n_t}), J(z_n(t))\rangle \\
&\qquad + \langle F(t+h, u_{n_t}) - F(t, u_{n_t}), J(z_n(t))\rangle \\
&\quad \leqslant hL_1(\|u_n(t)\|)(1 + \|A_n(t)\,u_n(t)\|)\,\|z_n(t)\| \\
&\qquad + B\,\|u_{n_{t+h}} - u_{n_t}\|_C\,\|z_n(t)\| + hg(\|u_{n_t}\|_C)\,\|z_n(t)\| \qquad (2.14)
\end{aligned}$$

a.e. in $[0, T]$. Here we have used the accretiveness of $A_n(t)$, (2.3), (C.5), and (C.6).

In Lemma 2.3 we have established that there exists $K' > 0$ such that $\|u_{n_t} - \phi\|_C \leqslant K'$, $\|u_{n_t}\|_C \leqslant K'$, and $\|u_n(t)\| \leqslant K'$. Since

$$\begin{aligned}
\|A_n(t)\,u_n(t)\| &\leqslant \|u_n'(t)\| + \|F(t, u_{n_t})\| \\
&\leqslant \|u_n'(t)\| + B\,\|u_{n_t} - \phi\|_C + \|F(t, \phi)\| \\
&\leqslant \|u_n'(t)\| + BK' + M_1,
\end{aligned}$$

for some $M_1 > 0$, by the continuity of F in t, (2.14) yields

$$(d/dt)\,\|z_n(t)\| \leqslant hC_1 + hC_2\,\|u_n'(t)\| + B\,\|u_{n_{t+h}} - u_{n_t}\|_C,$$

where $C_1 = L_1(K')(1 + BK' + M_1) + g(K')$, $C_2 = L_1(K')$. An integration above gives

$$\|z_n(t)\| \leqslant \|z_n(0)\| + hC_1 T + hC_2 \int_0^t \|u_n'(s)\|\,ds$$

$$+ B \int_0^t \|u_{n_{s+h}} - u_{n_s}\|_C\,ds,$$

or

$$\begin{aligned}
&\|u_n(t+h) - u_n(t)\|/h \\
&\quad \leqslant \|u_n(h) - u_n(0)\|/h + C_1 T + C_2 \int_0^t \|u_n'(s)\|\,ds \\
&\qquad + B \int_0^t \|u_{n_{s+h}} - u_{n_s}\|_C/h\,ds. \qquad (2.15)
\end{aligned}$$

Case 1. Suppose that $t+h<r$; that is, $t<r$ and h is sufficiently small so that $t+h<r$. Then we have

$$\begin{aligned}
&\int_0^t \|u_{n_{s+h}}-u_{n_s}\|_C/h\,ds \\
&\quad=\int_0^t \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds \\
&\quad\leqslant\int_0^t \sup_{\theta\in[-r,-(s+h))} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds \\
&\qquad+\int_0^t \sup_{\theta\in[-(s+h),-s]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds \\
&\qquad+\int_0^t \sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds. \qquad (2.16)
\end{aligned}$$

If $\theta\in[-r,-(s+h))$, then $s+h+\theta<0$ and $s+\theta<0$. For such θ's,

$$\begin{aligned}
&\|u_n(s+h+\theta)-u_n(s+\theta)\|/h \\
&\quad=\|\phi(s+h+\theta)-\phi(s+\theta)\|/h\leqslant N_1 h/h=N_1 \qquad (2.17)
\end{aligned}$$

for some $N_1>0$, by our assumption that $\phi'\in C$.

If $\theta\in[-(s+h),-s]$, then $s+h+\theta\geqslant 0$ but $s+\theta\leqslant 0$. Since $u_n\in C^1([0,T],X)$ and $\phi\in C^1([-r,0],X)$, Lemma 2.5 implies

$$\overline{\lim_{h\to 0^+}}\ \sup_{\theta\in[-(s+h),-s]} \|u_n(s+\theta+h)-u_n(s+\theta)\|/h\leqslant\|u_n'(0)\|+\|\phi'(0)\|, \quad (2.18)$$

where $u_n'(0)$ is the right derivative of u_n at 0, and $\phi'(0)$ is the left derivative of ϕ at 0. If $\theta\in[-s,0]$, then $s+\theta\geqslant 0$, and for such θ's, by Lemma 2.4,

$$\begin{aligned}
&\lim_{h\to 0^+}\ \sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h \\
&\quad=\sup_{\theta\in[-s,0]}\ \lim_{h\to 0^+} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h \\
&\quad=\sup_{\theta\in[-s,0]} \|u_n'(s+\theta)\|. \qquad (2.19)
\end{aligned}$$

Thus, for each $t\in[0,r)$, (2.15)–(2.19), and Fatou's lemma imply

$$\begin{aligned}
\overline{\lim_{h\to 0+}} &\|u_n(t+h)-u_n(t)\|/h \\
&= \lim_{h\to 0+} \|u_n(t+h)-u_n(t)\|/h \\
&\leqslant \|u_n'(0)\| + C_1 T + C_2 \int_0^t \|u_n'(s)\|\,ds + B\int_0^t N_1\,ds \\
&\quad + B\int_0^t (\|u_n'(0)\| + \|\phi'(0)\|)\,ds \\
&\quad + B\int_0^t \sup_{\theta\in[-s,0]} \|u_n'(s+\theta)\|\,ds.
\end{aligned}$$

Since

$$\begin{aligned}
\|u_n'(s)\| &= \lim_{h\to 0+} \|u_n(s+h)-u_n(s)\|/h \\
&\leqslant \lim_{h\to 0+} \sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h,
\end{aligned}$$

we have

$$\begin{aligned}
\lim_{h\to 0+} &\|u_n(t+h)-u_n(t)\|/h \\
&= \|u_n'(t)\| \leqslant (1+BT)\|u_n'(0)\| + [C_1 + B(N_1 + \|\phi'(0)\|)]T \\
&\quad + (C_2+B)\int_0^t \sup_{\theta\in[-s,0]} \|u_n'(s+\theta)\|\,ds. \qquad (2.20)
\end{aligned}$$

Case 2. Suppose $t \geqslant r$.
Then, for $r \leqslant t+h \leqslant T$, we obtain

$$\begin{aligned}
\int_0^t &\|u_{n_{s+h}} - u_{n_s}\|_C/h\,ds \\
&= \int_0^t \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds \\
&= \int_0^{r-h} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds \\
&\quad + \int_{r-h}^t \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds.
\end{aligned}$$

It follows that

$$\int_0^t \|u_{n_{s+h}} - u_{n_s}\|_C \, ds$$

$$\leqslant \int_0^{r-h} \sup_{\theta \in [-r, -(s+h))} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds$$

$$+ \int_0^{r-h} \sup_{\theta \in [-(s+h), -s]} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds$$

$$+ \int_0^{r-h} \sup_{\theta \in [-s, 0]} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds$$

$$\int_{r-h}^t \sup_{\theta \in [-r, 0]} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds. \tag{2.21}$$

If $\theta \in [-r, -(s+h))$ with $s+h \leqslant r$, then $s+h+\theta < 0$, and $s+\theta < 0$. For such θ's we obtain, as in Case 1,

$$\|u_n(s+h+\theta) - u_n(s+\theta)\|/h \leqslant N_1,$$

so that

$$\int_0^{r-h} \sup_{\theta \in [-r, -(s+h))} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds \leqslant N_1 T. \tag{2.22}$$

For $\theta \in [-(s+h), -s]$ we have $s+h+\theta \geqslant 0$, but $s+\theta \leqslant 0$. For such θ's, as in Case 1, we find

$$\overline{\lim_{h \to 0^+}} \sup_{\theta \in [-(s+h), -s]} \|u_n(s+\theta+h) - u_n(s+\theta)\|/h \leqslant \|u_n'(0)\| + \|\phi'(0)\|. \tag{2.23}$$

Since

$$\int_0^{r-h} \sup_{\theta \in [-s, 0]} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds$$

$$\leqslant \int_0^t \sup_{\theta \in [-s, 0]} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds$$

$$- \int_r^t \sup_{\theta \in [-s, 0]} \|u_n(s+h+\theta) - u_n(s+\theta)\|/h \, ds,$$

and

$$\begin{aligned}
&\int_{r-h}^{t} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds\\
&\quad=\int_{r}^{t} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds\\
&\qquad+\int_{r-h}^{r} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds\\
&\quad\leqslant\int_{r}^{t} \sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds\\
&\qquad+\int_{r-h}^{r} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds,
\end{aligned}$$

we arrive at

$$\begin{aligned}
&\int_{0}^{r-h} \sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds\\
&\qquad+\int_{r-h}^{t} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds\\
&\quad\leqslant\int_{0}^{t} \sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds\\
&\qquad+\int_{r-h}^{r} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds. \qquad (2.24)
\end{aligned}$$

We now show that

$$\lim_{h\to 0^+}\int_{r-h}^{r} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\,ds=0.$$

Since $-s-h\leqslant -r$, the integrand satisfies

$$\begin{aligned}
&\sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\\
&\quad\leqslant \sup_{\theta\in[-s-h,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\\
&\quad\leqslant \sup_{\theta\in[-s-h,-s]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\\
&\qquad+\sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h. \qquad (2.25)
\end{aligned}$$

From the proof of Lemma 2.5 we see that

$$\sup_{\theta\in[-(s+h),-s]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h \leqslant \|u_n'(\mu h)\| + \|\phi'(-\gamma h)\|,$$

where $\mu \in (0, 1)$, and $\gamma \in (0, 1)$. Thus,

$$\begin{aligned}
&\lim_{h\to 0^+} \int_{r-h}^{r} \sup_{\theta\in[-s-h,-s]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\, ds \\
&\quad\leqslant \lim_{h\to 0^+} \int_{r-h}^{r} (\|u_n'(\mu h)\| + \|\varphi'(-\gamma h)\|)\, ds \\
&\quad= \lim_{h\to 0^+} [\|u_n'(\mu h)\| + \|\phi'(-\gamma h)\|]h \\
&\quad= 0.
\end{aligned} \tag{2.26}$$

Also, for some $\theta_1 \in [-s, 0]$, $\lambda \in (0, 1)$,

$$\begin{aligned}
\sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h &= \|u_n(s+h+\theta_1)-u_n(s+\theta_1)\|/h \\
&\leqslant \|u_n'(s+\theta_1+\lambda h)\| \\
&\leqslant \sup_{0\leqslant s\leqslant T} \|u_n'(s)\| = N_2(n).
\end{aligned}$$

In view of this appraisal, we obtain

$$\begin{aligned}
&\lim_{h\to 0^+} \int_{r-h}^{r} \sup_{\theta\in[-s,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\, ds \\
&\quad\leqslant \lim_{h\to 0^+} N_2(n)h = 0.
\end{aligned} \tag{2.27}$$

Hence, from (2.25)–(2.27) we get

$$\lim_{h\to 0^+} \int_{r-h}^{r} \sup_{\theta\in[-r,0]} \|u_n(s+h+\theta)-u_n(s+\theta)\|/h\, ds = 0. \tag{2.28}$$

Now we use (2.21)–(2.24), (2.28), and Lemma 2.4 to obtain

$$\begin{aligned}
&\overline{\lim_{h\to 0^+}} \int_0^t \|u_{n_{s+h}} - u_{n_s}\|_C\, ds \\
&\quad\leqslant N_1 T + (\|u_n'(0)\| + \|\phi'(0)\|)T + \int_0^t \sup_{\theta\in[-s,0]} \|u_n'(s+\theta)\|\, ds.
\end{aligned} \tag{2.29}$$

It follows that for $t \geqslant r$, (2.15) and (2.29) give

$$\begin{aligned}
&\overline{\lim_{h\to 0+}} \|u_n(t+h)-u_n(t)\|/h \\
&\quad = \lim_{h\to 0+} \|u_n(t+h)-u_n(t)\|/h = \|u_n'(t)\| \\
&\quad \leqslant (1+BT)\|u_n'(0)\| + [C_1 + B(N_1 + \|\phi'(0)\|)]T \\
&\quad\quad + (C_2+B)\int_0^t \sup_{\theta\in[-s,0]} \|u_n'(s+\theta)\|\, ds. \qquad (2.30)
\end{aligned}$$

Taking into consideration (2.20), we see that (2.30) holds for all $t \in [0, T]$. From (2.30) and

$$\|u_n'(0)\| \leqslant \|A_n(0)\, u_n(0)\| + \|F(0, \phi)\| \leqslant \|A(0)a\| + M_1,$$

we conclude that for any $\theta \leqslant 0$ such that $t+\theta \in [0, T]$, we have

$$\lim_{h\to 0+} \|u_n(t+\theta+h) - u_n(t+\theta)\|/h \leqslant C_3 + C_4 \int_0^t \sup_{\theta\in[-s,0]} \|u_n'(s+\theta)\|\, ds,$$

where C_3, C_4 are appropriate constants. Hence,

$$\begin{aligned}
&\sup_{\theta\in[-t,0]} \lim_{h\to 0+} \|u_n(t+\theta+h) - u_n(t+\theta)\|/h \\
&\quad = \sup_{\theta\in[-t,0]} \|u_n'(t+\theta)\| \leqslant C_3\, e^{C_4 T} = N
\end{aligned}$$

by Gronwall's inequality. The uniform boundedness of $\{u_n'(t)\}$ follows from

$$\|u_n'(t)\| \leqslant \sup_{\theta\in[-t,0]} \|u_n'(t+\theta)\|.$$

LEMMA 2.7. *Assume that the conditions of Lemma 2.3 hold. Then the strong limit $u(t) = \lim_{n\to\infty} u_n(t)$ exists uniformly on $[0, T]$.*

Proof. Let $x_{mn}(t) = u_m(t) - u_n(t)$. Then we have, a.e.,

$$\begin{aligned}
(1/2)(d/dt)\|x_{mn}(t)\|^2 &= \langle x_{mn}'(t), J(x_{mn}(t))\rangle \\
&= -\langle A_m(t)\, u_m(t) - A_n(t)\, u_n(t), J(x_{mn}(t))\rangle \\
&\quad + \langle F(t, u_{m_t}) - F(t, u_{n_t}), J(x_{mn}(t))\rangle. \qquad (2.31)
\end{aligned}$$

Since $A_m(t)\, u_m(t) = A(t)\, J_m(t)\, u_m(t)$, $A_n(t) = A(t)\, J_n(t)\, u_n(t)$, and $A(t)$ is accretive,

$$\langle A_m(t)\, u_m(t) - A_n(t)\, u_n(t), J(y_{mn}(t))\rangle \geqslant 0, \qquad (2.32)$$

where $y_{mn}(t) = J_m(t)\, u_m(t) - J_n(t)\, u_n(t)$. Adding (2.31) and (2.32), we get

$$\begin{aligned}(1/2)(d/dt)\,\|x_{mn}(t)\|^2 &\leqslant \langle A_m(t)\, u_m(t) - A_n(t)\, u_n(t), J(y_{mn}(t)) - J(x_{mn}(t))\rangle \\ &\quad + \langle F(t, u_{m_t}) - F(t, u_{n_t}), J(x_{mn}(t))\rangle \\ &\leqslant \|A_m(t)\, u_m(t) - A_n(t)\, u_n(t)\|\, \|J(y_{mn}(t)) - J(x_{mn}(t))\| \\ &\quad + B\, \|u_{m_t} - u_{n_t}\|_C\, \|u_m(t) - u_n(t)\|\end{aligned}$$

a.e. By the uniform boundedness of $\{u_n(t)\}$ and $\{u_n'(t)\}$ (Lemmas 2.3 and 2.6),

$$\begin{aligned}\|A_n(t)\, u_n(t)\| &\leqslant \|u_n'(t)\| + \|F(t, u_{n_t})\| \\ &\leqslant \|u_n'(t)\| + B\, \|u_{n_t} - \phi\|_C + \|F(t, \phi)\| \\ &\leqslant N + BK' + M_1 = M_0. \end{aligned} \tag{2.33}$$

From the absolute continuity of $\|x_{mn}(t)\|^2$, and the fact that $x_{mn}(0) = 0$, we obtain

$$\begin{aligned}\|x_{mn}(t)\|^2 &= \|u_m(t) - u_n(t)\|^2 \\ &\leqslant 4M_0 \int_0^T \|J(y_{mn}(s)) - J(x_{mn}(s))\|\, ds + 2B \int_0^t \|u_{m_s} - u_{n_s}\|_C^2\, ds.\end{aligned}$$

As in the proof of Lemma 2.3, this implies that

$$\begin{aligned}\|u_{m_t} - u_{n_t}\|_C^2 \leqslant 4M_0 &\int_0^T \|J(y_{mn}(s)) - J(x_{mn}(s))\|\, ds \\ &+ 2B \int_0^t \|u_{m_s} - u_{n_s}\|_C^2\, ds,\end{aligned}$$

from which we conclude (by Gronwall's inequality) that

$$\|u_{m_t} - u_{n_t}\|_C^2 \leqslant 4M_0\, e^{2BT} \int_0^T \|J(y_{mn}(s)) - J(x_{mn}(s))\|\, ds. \tag{2.34}$$

We now show that $\|u_{m_t} - u_{n_t}\|_C \to 0$ uniformly in t, by showing that the integrand in (2.34) converges to zero uniformly in s as $m, n \to \infty$. We first observe that $\|x_{mn}(t)\| = \|u_m(t) - u_n(t)\| \leqslant 2K'$. Also, by (2.2) and (2.33),

$$\begin{aligned}\|y_{mn}(t) - x_{mn}(t)\| &\leqslant \|J_m(t)(u_m(t)) - u_m(t)\| + \|J_n(t)(u_n(t)) - u_n(t)\| \\ &\leqslant (1/m)\, \|A_m(t)\, u_m(t)\| + (1/n)\, \|A_n(t)\, u_n(t)\| \\ &\leqslant [(m+n)/mn] M_0,\end{aligned}$$

which tends to zero as $m, n \to \infty$. By the uniform continuity of J on bounded subsets of X, we have that given $\varepsilon > 0$, $\|J(y_{mn}(t)) - J(x_{mn}(t))\| < \varepsilon$, $0 \leqslant t \leqslant T$, for all sufficiently large m, n. Thus,

$$\lim_{n \to \infty} u_{n_t} = u_t \text{ exists uniformly in } t.$$

Since

$$\|u_m(t) - u_n(t)\| \leqslant \sup_{\theta \in [-r, 0]} \|u_m(t+\theta) - u_n(t+\theta)\| = \|u_{m_t} - u_{n_t}\|_C,$$

the above limit implies that $\|u_m(t) - u_n(t)\| \to 0$ uniformly in t as $m, n \to \infty$. This implies in turn that

$$\lim_{n \to \infty} u_n(t) = u(t) \text{ exists uniformly in } t.$$

Since $u_n(t)$ is Lipschitz continuous with Lipschitz constant independent of n ($\|u_n'(t)\| \leqslant N$), the limit $u(t)$ is also Lipschitz continuous with $u(0) = \phi(0) = a$. From $u_{n_t} \to u_t$ we also conclude that $u_0 = \phi$.

The proof of the following lemma follows as in Kato's paper [13], and is therefore omitted.

LEMMA 2.8. *Let the conditions of Lemma 2.3 hold. If $u(t) = \lim_{n \to \infty} u_n(t)$ (Lemma 2.7), then $u(t) \in D$ for all $t \in [0, T]$, and $A(t)\, u(t)$ is bounded and weakly continuous. Moreover, the function $-A(t)\, u(t) + F(t, u_t)$ is Bochner integrable and $u(t)$ is an indefinite integral of $-A(t)\, u(t) + F(t, u_t)$. The strong derivative $u'(t)$ also exists a.e., and equals $-A(t) u(t) + F(t, u_t)$.*

The Lipschitz continuity of $u(t)$ with respect to the initial function is shown in the following lemma.

LEMMA 2.9. *There exists a constant $C_0 > 0$ with the following property: let $u(t)$, $v(t)$ be two solutions satisfying the conclusion of Lemma 2.8, with initial conditions $u_0 = \phi \in C$, $v_0 = \psi \in C$, respectively. Then we have*

$$\|u(t) - v(t)\| \leqslant C_0 \|\phi - \psi\|_C, \qquad t \in [0, T].$$

Proof. The function $x(t) = u(t) - v(t)$ has weak derivative which is weakly continuous, and hence bounded. Thus, $x(t)$ is Lipschitz continuous. It follows that $\|x(t)\|$ is differentiable a.e. Consequently we have

$$\begin{aligned}(1/2)(d/dt)\,\|x(t)\|^2 &= -\langle A(t)\,u(t) - A(t)\,v(t), J(x(t))\rangle \\ &\quad + \langle F(t, u_t) - F(t, v_t), J(x(t))\rangle \\ &\leqslant \langle F(t, u_t) - F(t, v_t), J(x(t))\rangle \\ &\leqslant B\,\|u_t - v_t\|_C^2\end{aligned}$$

a.e. An integration of this inequality gives

$$\|x(t)\|^2 = \|u(t) - v(t)\|^2 \leqslant \|\phi(0) - \psi(0)\|^2 + B\int_0^t \|u_s - v_s\|_C^2\,ds.$$

As in the proof of Lemma 2.3, we conclude that

$$\|u_t - v_t\|_C^2 \leqslant \|\phi - \psi\|_C^2 + B\int_0^t \|u_s - v_s\|_C^2\,ds,$$

which yields (by Gronwall's inequality)

$$\|u_t - v_t\|_C^2 \leqslant \|\phi - \psi\|_C^2\,e^{BT}.$$

Thus, $\|u(t) - v(t)\| \leqslant e^{BT/2}\,\|\phi - \psi\|_C$.

The proof of Theorem 2.1 has now been accomplished.

3. An Example

As an illustration of how the results of Section 2 may be applied, we consider the following nonlinear functional heat equation:

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t} - a(t)k\left(\frac{\partial u(x,t)}{\partial x}\right)\frac{\partial^2 u(x,t)}{\partial x^2} \\ &\qquad = f(t, u(x, t-r)), \qquad r > 0,\ t \in [0, T],\ x \in (0, 1),\end{aligned} \tag{3.1}$$

with the boundary conditions:

$$\begin{aligned}u(x, \theta) &= \phi(x, \theta), && -r \leqslant \theta \leqslant 0,\\ u(0, t) &= \alpha u'(0, t), && 0 \leqslant t \leqslant T,\\ u(1, t) &= -\beta u'(1, t), && 0 \leqslant t \leqslant T.\end{aligned} \tag{B}$$

Here, a, k, f are three given functions with a: $[0, T] \to [m, \infty)$ (for some constant $m > 0$) Lipschitzian, $k\colon R \to R$ continuous, and f: $[0, T] \times R \to R$ globally Lipschitzian, jointly in its two variables. The autonomous ($a(t) \equiv 1$), homogeneous version of ((3.1), (B)) represents a problem in heat conduction, where the thermal conductivity depends on the temperature gradient (the function k depends on the thermal conductivity), and has been studied by Burch and Goldstein in [3].

Now let $X = L^p[0, 1]$, for some $p \in (1, \infty)$, and let

$$D = \{u \in L^p[0, 1]: u \in C^2[0, 1], u(0) = \alpha u'(0), u(1) = -\beta u'(1)\}.$$

Now define the operator $\tilde{A}(t)$: $D \subset X \to X$ as follows:

$$\tilde{A}(t)\, u(x) = -a(t)k\left(\frac{du(x)}{dx}\right)\frac{d^2u(x)}{dx^2}$$

for every $t \in [0, T]$ and $x \in (0, 1)$. For k bounded below by a positive constant and $\alpha \geqslant 0$, $\beta \geqslant 0$ (or for k continuously differentiable, positive and $\alpha > 0$, $\beta > 0$), Burch and Goldstein [3, Theorem 2.2] have shown that the closure A of $\tilde{A}$ with $a(t) \equiv 1$ is m-accretive for all $p \in [1, \infty)$. Since $a(t) > 0$, our corresponding operator $A(t)$ is also m-accretive for such functions k. Moreover, for some constant $B > 0$,

$$\begin{aligned} \|\tilde{A}(t)u - \tilde{A}(s)u\| &= |a(t) - a(s)| \left\| k\left(\frac{du}{dx}\right)\frac{d^2u}{dx^2} \right\| \\ &\leqslant B\,|t-s|\,\|\tilde{A}(t)u\|/a(t) \\ &\leqslant (B/m)\,|t-s|\,\|\tilde{A}(t)u\| \\ &\leqslant (B/m)\,|t-s|\,(1 + \|\tilde{A}(t)u\|). \end{aligned}$$

Thus, we see that Conditions (C.1)–(C.4) are satisfied. Let $C = C([-r, 0], X)$ with the supremum norm. For $\phi \in C$, we define $F(t, \phi) = f(t, \phi(-r))$. Then, clearly, F satisfies (C.5) and (C.6). We can now write the problem ((3.1), (B)) in the form (E), and conclude, by virtue of Theorem 2.1, the existence of a unique solution.

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